Lance Chapter 2: Multipliers and morphisms

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1 Reminder of some definitions

Definition 1.1. Throughout this section A will be a C*-algebra and X will be a *(right) Hilbert A-module*. That is, X is a complex vector space with a (right) A-action, i.e., a bilinear pairing $(x, a) \mapsto x \cdot a : \mathsf{X} \times A \to \mathsf{X}$ satisfying:

(i) $(x + y) \cdot a = x \cdot a + y \cdot a$,

(ii) $x \cdot (a + b) = x \cdot a + x \cdot b$,

(iii) $(x \cdot a) \cdot b = x \cdot (ab)$,

(iv) $(\lambda x) \cdot a = x \cdot (\lambda a) = \lambda (x \cdot a)$.

In addition, X has an A-valued the inner product $\langle \cdot, \cdot \rangle_A : \mathsf{X} \times \mathsf{X} \to A$ satisfying

(i) $\langle x, \lambda y + \mu z \rangle_A = \lambda \langle x, y \rangle_A + \mu \langle x, z \rangle_A$,

(ii) $\langle x, y \cdot a \rangle_A = \langle x, y \rangle_A a$,

(iii) $\langle x, y \rangle_A^* = \langle y, x \rangle_A$,

(iv) $\langle x, x \rangle_A \geq 0$ (as a positive element of A),

(v) $\langle x, x \rangle = 0$ implies that $x = 0$.

Moreover, X is complete with respect to the norm defined by $\|x\|_A := \|\langle x, x \rangle_A\|^{1/2}$.

Definition 1.2. A Hilbert A-module is *full* if the ideal

$$I = \operatorname{span}\{\langle x, y \rangle_A : x, y \in \mathsf{X}\}$$

is dense in A.

Definition 1.3. Suppose X and Y are Hilbert A-modules. A function $T : \mathsf{X} \to \mathsf{Y}$ is *adjointable* if there is a function $T^* : \mathsf{Y} \to \mathsf{X}$ such that

$$\langle T(x), y \rangle_A = \langle x, T^*(y) \rangle_A \quad \text{for all } x, y \in \mathsf{X}$$

We denote by $\mathcal{L}(\mathsf{X}, \mathsf{Y})$ the set of all adjointable operators from X to Y, and $\mathcal{L}(\mathsf{X})$ for $\mathcal{L}(\mathsf{X}, \mathsf{X})$.

Definition 1.4. Let X and Y be two Hilbert A-modules. For $x \in \mathsf{X}$ and $y \in \mathsf{Y}$, we define $\theta_{x,y} : \mathsf{X} \to \mathsf{Y}$ by $\theta_{x,y}(z) = x \cdot \langle y, z \rangle_A$. We define $\mathcal{K}(\mathsf{X}, \mathsf{Y})$ to be the closed linear subspace of $\mathcal{L}(\mathsf{X}, \mathsf{Y})$ spanned by $\{\theta_{x,y} : x \in \mathsf{X} \text{ and } y \in \mathsf{Y}\}$. In particular, $\mathcal{K}(\mathsf{X}) := \mathcal{K}(\mathsf{X}, \mathsf{X})$.

Definition 1.5. In the *strict topology (strong operator topology)* on $\mathcal{L}(\mathsf{X})$, $T_i \to T$ if and only if $T_i x \to Tx$ (in norm topology on X) for every $x \in \mathsf{X}$.

2 Unitization

There are more than one way to embed a C*-algebra in a unital C*-algebra.

Definition 2.1. An ideal I in a C*-algebra A is *essential* if I has nonzero intersection with every other nonzero ideal in A.

Lemma 2.2. *An ideal I is essential if and only if $aI = \{0\}$ implies $a = 0$.*

Theorem 2.3. *Up to isomorphism, there is a unique unital C*-algebra which contains A as an essential ideal and is maximal in the sense that any other such algebra can be embedded in it.*

This C*-algebra is called the multiplier algebra and is denoted by $M(A)$.

Definition 2.4. *(via Universal Property)* Let A be a C*-algebra. Its multiplier algebra $M(A)$ is any C*-algebra satisfying the following universal property: for all C*-algebra D containing A as an essential ideal, there exists a unique $*$-homomorphism $\Psi : D \to M(A)$ such that Ψ extends the identity homomorphism on A.

2.1 Double centralizers (from *Murphy*)

The "traditional" definition of $M(A)$ for any C*-algebra A is given in terms of the double centralizer.

Definition 2.5. A double centralizer for a C*-algebra A is a pair (L, R) of bounded linear maps on A, such that for all $a, b \in A$

$$L(ab) = L(a)b, \qquad R(ab) = aR(b) \quad \text{and} \quad R(a)b = aL(b).$$

Example 2.6. For example, if $c \in A$ is fixed, and L_c, R_c are the linear maps on A defined by $L_c(a) = ca$ and $R_c(a) = ac$, then (L_c, R_c) is a double centralizer on A.

In Example 2.6, one can check that $\|L_c\|_{\text{op}} = \|R_c\|_{\text{op}} = \|c\|$. The map $c \mapsto (L_c, R_c)$ ends up being how we embed A into $M(A)$. More generally, we have the following:

Lemma 2.7. *If (L, R) is a double centralizer on a C*-algebra A, then $\|L\| = \|R\|$.*

Denote the set of all double centralizers on a C*-algebra A by $M(A)$. We define the norm on (L, R) to be $\|L\| = \|R\|$ from Lemma 2.7. We define the following operators on $M(A)$:

(i) $(L_1, R_1) + (L_2, R_2) = (L_1 + L_2, R_2 + R_1)$,

(ii) $\lambda(L, R) = (\lambda L, \lambda R)$,

(iii) $(L_1, R_1)(L_2, R_2) = (L_1 L_2, R_2 R_1)$,

(iv) $(L, R)^* = (R^*, L^*)$

Theorem 2.8. *If A is a C*-algebra, then $M(A)$ with the norm and operations defined as above is a unital C*-algebra with identity* $(\mathrm{id}_A, \mathrm{id}_A)$.

The map $A \to M(A)$, $a \mapsto (L_a, R_a)$ is an isometric $*$-homomorphism, so A can be realized as a closed $*$-subalgebra of $M(A)$.

2.2 $M(A)$ as the Adjointable Operators on A_A

For any C*-algebra A, we can form a (right) Hilbert A-module, $\mathsf{X} = A_A$, by letting A act on itself by right multiplication: $a \cdot b = ab$, and define $\langle a, b \rangle_A = a^* b$.

We identify A with $\mathcal{K}(A_A)$ in the following way. For each fixed $a \in A$, we can consider the (adjointable) operators L_a on A_A given by $L_a(x) = ax$, whose adjoint is L_{a^*}. One checks that the map

$$L : \begin{cases} A \to \mathcal{L}(A_A) \\ a \mapsto L_a \end{cases}$$

is an isometric homomorphism, so it embeds A onto a C*-subalgebra of $\mathcal{L}(A_A)$. Since

$$\theta_{a,b}(c) = a\langle b, c \rangle_A = ab^* c = L_{ab^*}(c)$$

and products of the form ab^* is dense in A, this map L is an isomorphism of A onto $\mathcal{K}(A_A)$.

It will turn out that $M(A)$ can be identified with $\mathcal{L}(A_A)$. To show this, we need to prove the following:

- $A \cong \mathcal{K}(A_A)$ is an essential ideal of $\mathcal{L}(A_A)$. This follows from Lemma 2.2.
- maximal: any C*-algebra B containing A as an essential ideal embedded in $\mathcal{L}(A_A)$,
- $\mathcal{L}(A_A)$ is unique.

Definition 2.9. Let A be a C*-algebra and X be a Hilbert A-module. We say that a $*$-homomorphism is *nondegenerate* if $\alpha(A)\mathsf{X}$ (linear space of products of elements of $\alpha(A)$ and X) is dense in X.

Example 2.10. The inclusion map $\mathcal{K}(\mathsf{X}) \hookrightarrow \mathcal{L}(\mathsf{X})$ is nondegenerate.

Proposition 2.11. *Let A, B, and C be C*-algebras with B an ideal in C, and let X be a Hilbert A-module. If $\alpha : B \to \mathcal{L}(\mathsf{X})$ is a nondegenerate $*$-homomorphism, then α extends uniquely to a $*$-homomorphism $\overline{\alpha} : C \to \mathcal{L}(\mathsf{X})$. If α is injective and $B$is essential in C, then $\overline{\alpha}$ is injective.*

Applying to $B = A$ and $\mathsf{X} = A_A$ with α being the embedding of A in $\mathcal{L}(A_A)$, it follows that any C*-algebra containing A as an essential ideal can be embedded in $\mathcal{L}(A_A)$. Uniqueness also follows fro the above Proposition.

Example 2.12 (Theorem). For a Hilbert A-module X, $\mathcal{L}(X) \cong M(\mathcal{K}(\mathsf{X}))$.

Example 2.13. Let X be a locally compact space, then $M(C_0(X)) \cong C(\beta X)$, where βX is the Stone-Čech compactification of X.

3 Morphisms

Definition 3.1. For C*-algebras A and B, we define a morphism from A to B to be a nondegenerate $*$-homomorphism from A to $M(B)$. We denote the set of all morphisms from A to B by $\mathrm{Mor}(A, B)$.

Proposition 3.2. *Let A and B be C*-algebras and let X be a Hilbert B-module. For a $*$-homomorphism $\alpha : A \to \mathcal{L}(\mathsf{X})$, the following conditions are equivalent:*

(i) α *is nondegenerate,*

(ii) α *is the restriction of* A *of a unital* $*$*-homomorphism* $\overline{\alpha}$ *from* $M(A)$ *to* $\mathcal{L}(\mathsf{X})$ *which is strictly continuous on the unit ball,*

(iii) for some approximate unit (e_i) *of* A*,* $\alpha(e_i) \to \mathrm{Id}_{\mathsf{X}}$ *strictly, where* Id_{X} *is the identity map on* X*.*

To ease notation, we usually just write $\alpha \in \mathrm{Mor}(A, B)$ (rather than $\overline{a}$) to denote the extension of α to $M(A)$. For $\beta \in Mor(B, C)$, the composition defined when we interpret β to be its extension from $M(B)$ to $M(C)$.